Appl Phys B
DOI 10.1007/s00340-011-4661-6

Applied Physics B
Lasers and Optics

Intrinsic axes of partially coherent light beams and their invariance through rotationally symmetric ABCD optical systems

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Received: 20 January 2011 / Revised version: 12 May 2011

Abstract New transverse orthogonal axes associated with the beam shape are introduced on the basis of the irradiance-moment formalism that describes the overall spatial structure of partially coherent paraxial beams. We refer to them as the intrinsic axes of the beam. The angle between these axes and the laboratory reference coordinates can be determined from the measurements of the second-order spatial moments of the beam, even although the field amplitude is not analytically known. It has been found that the intrinsic axes remain invariant upon propagation through rotationally symmetric first-order optical systems. Two additional analytical quantities have also been introduced. One of them provides a measure of the beam symmetry, and the other one contains information about the beam rotation under propagation. When the field is referred to the intrinsic axes, these two quantities are shown to reach their extreme values. The results are illustrated by means of an example.

1 Introduction

The use of overall measurable parameters for describing the spatial behavior of light fields is a practical tool in laser beam propagation. In particular, the existence of invariants can provide intrinsic properties of the beams. As is well known [1–14], the global spatial structure of paraxial fields can be characterized by their irradiance moments. Thus, the beam width, the far-field divergence, the beam quality (also called beam propagation factor) and the orbital angular momentum have been formulated on the basis of the second-order moments. Moreover, some of these parameters represent current ISO standards for beam characterization [15].

Such spatial description of the transverse beam profile should be expressed with respect to Cartesian coordinate axes. Defined in terms of the beam moments, several reference systems have been reported in the literature (see, for example, [15, 16]). Along the directions of these specific coordinate axes, it has also been shown that either the beam width or the far-field divergence reaches their extreme values. However, these reference systems change under propagation through rotationally symmetric first-order optical (RSO) systems.

In the present work, on the basis of the second-order moments formalism, new Cartesian coordinate axes are introduced, which remain invariant when the beam travels through RSO systems. We refer to these axes as the intrinsic axes of the beam. The paper is then organized as follows. In the next section, the basic formalism is shown. Section 3 analytically defines the intrinsic axes of a partially coherent (paraxial) beam. In Sect. 4, two quantities are provided that reach their extreme values along the intrinsic axes. The physical meaning of these quantities is discussed, and the angle between the intrinsic axes and the laboratory reference axes is also given. In the same section, the results are applied to an illustrative example. Finally, the main conclusions are summarized in Sect. 5.

2 Formalism

Within the framework of the paraxial approach, let us consider a quasimonochromatic partially coherent beam. Since

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Published online: 29 July 2011

we are interested in the overall spatial description of the field, we next introduce the well-known beam irradiance moments (denoted by sharp brackets), defined in terms of the Wigner distribution function, $h(r, \eta, z)$, associated to the field, in the form [3]

$$\langle x^m y^n u^p v^q \rangle \equiv \frac{1}{I_0} \int_{-\infty}^{\infty} x^m y^n u^p v^q h(\boldsymbol{r}, \boldsymbol{\eta}, z)\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{\eta}, \tag{1}$$

where m, n, p and q are integer numbers, $\boldsymbol{r} = (x, y)$ denotes the two-dimensional position vector transverse to the propagation direction z, $k\boldsymbol{\eta} = (ku, kv) = (k_x, k_y)$ provides the wavevector components along the x- and y-laboratory axes (accordingly, u and v represent angles of propagation, without taking the evanescent waves into account) and $I_o = \int h\, \mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{\eta}$ is proportional to the total beam power. The first-order moments, namely, $\langle x \rangle$, $\langle y \rangle$, $\langle u \rangle$ and $\langle v \rangle$, characterize the center of the beam and its mean direction. For the sake of simplicity, in what follows we assume that these moments vanish (this is not a true restriction, since it is equivalent to a shift of the Cartesian coordinate system). On the other hand, the (squared) beam width at a plane $z =$ constant and the (squared) far-field divergence are represented by $\langle x^2 + y^2 \rangle$ and $\langle u^2 + v^2 \rangle$, respectively. In addition, the crossed moment $\langle xu + yv \rangle$ gives the position of the beam waist through the condition $\langle xu + yv \rangle = 0$.

So far, the overall spatial structure of the irradiance profile has been characterized [13, 16] by two Cartesian coordinate systems:

(a) The orthogonal axes (frequently called principal axes) for which the crossed x–y moment vanishes, i.e., $\langle xy \rangle = 0$. Along such axes, we know that the beam widths $\langle x^2 \rangle^{1/2}$ and $\langle y^2 \rangle^{1/2}$ reach their extreme values [13, 16]. In general, the principal axes as well as the spatial profile rotate as the field propagates in free space. Accordingly, the principal axes are used to describe this rotation: it suffices to determine the angle ξ that the principal axes make with some fixed laboratory coordinate system. The result is

$$\tan(2\xi) = \frac{2\langle xy \rangle}{\langle x^2 \rangle - \langle y^2 \rangle}, \tag{2}$$

where the beam moments that appear in this expression are evaluated at plane $z = 0$.

(b) The orthogonal axes for which the crossed u–v moment vanishes, i.e., $\langle uv \rangle = 0$. Along such axes, it can be shown that the far-field divergences $\langle u^2 \rangle^{1/2}$ and $\langle v^2 \rangle^{1/2}$ reach their extreme values [16]. The angle ψ between these axes and some fixed laboratory coordinate axes is given by

$$\tan(2\psi) = \frac{2\langle uv \rangle}{\langle u^2 \rangle - \langle v^2 \rangle}, \tag{3}$$

where the beam moments are again evaluated at plane $z = 0$. It should be noted that these axes do not rotate upon free propagation. However, they change when the beam propagates through general rotationally symmetric ABCD systems.

We will next define certain transverse orthogonal axes that, in contrast to the previous coordinate systems, remain invariant when the beam travels through arbitrary RSO systems (including the free-propagation case).

3 Definition of intrinsic axes of a beam

For convenience, let us first introduce the following set of four 2×2 matrices [14, 17]:

$$\hat{M}_0 = \begin{pmatrix} \langle x^2 + y^2 \rangle & \langle xu + yv \rangle \\ \langle xu + yv \rangle & \langle u^2 + v^2 \rangle \end{pmatrix}, \tag{4a}$$

$$\hat{M}_1 = \begin{pmatrix} \langle x^2 - y^2 \rangle & \langle xu - yv \rangle \\ \langle xu - yv \rangle & \langle u^2 - v^2 \rangle \end{pmatrix}, \tag{4b}$$

$$\hat{M}_2 = \begin{pmatrix} 2\langle xy \rangle & \langle xv + yu \rangle \\ \langle xv + yu \rangle & 2\langle uv \rangle \end{pmatrix}, \tag{4c}$$

$$\hat{M}_3 = \begin{pmatrix} 0 & \langle xv - yu \rangle \\ \langle yu - xv \rangle & 0 \end{pmatrix}, \tag{4d}$$

We immediately see from (4a) that the near- and far-field behavior can be inferred from the diagonal elements of $\hat{M}_0$. Furthermore, the non-diagonal elements provide the position of the waist plane and its determinant gives the beam quality parameter (proportional to parameter M^2 introduced by Siegman [4]).

On the other hand, the matrix $\hat{M}_3$ informs us about the extrinsic (also called orbital) part of the angular momentum of the beam [12–14]. Finally, matrices $\hat{M}_1$ and $\hat{M}_2$ contain all the second-order parameters that describe the orientation of any beam profile upon free propagation.

It is also known [17] that matrices $\hat{M}_i$, $i = 0, 1, 2, 3$, propagate through RSO systems according to the simple law

$$(\hat{M}_i)_{\text{output}} = \hat{S}(\hat{M}_i)_{\text{input}}\hat{S}^t, \quad i = 0, 1, 2, 3, \tag{5}$$

where $\hat{S} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ denotes the 2×2 ABCD matrix representing the optical system, and t means transposition. Additional properties of this set of matrices can be found in [17].

In terms of these matrices $\hat{M}$, let us now introduce their "normalized" versions $\hat{m}_j$, $j = 1, 2$, namely,

$$\hat{m}_j = \hat{M}_0^{-1}\hat{M}_j, \quad j = 1, 2. \tag{6}$$

On this basis, we define the so-called *intrinsic axes* of a partially coherent beam as those orthogonal transverse axes fulfilling the condition

$$\mathrm{tr}(\hat{m}_1 \hat{m}_2) = 0, \tag{7}$$

where "tr" denotes the trace of the matrix. The name "intrinsic axes" refers to the fact that *they remain invariant (do not rotate) under propagation through RSO systems.* Let us show this.

Proof Note first that application of (5) enables us to propagate matrices $\hat{m}_j$. In fact, we have

$$(\hat{m}_j)_{\text{out}} = (\hat{M}_0)^{-1}_{\text{out}}(\hat{M}_j)_{\text{out}}$$
$$= (\hat{S}^t)^{-1}(\hat{M}_0)^{-1}_{\text{inp}}\hat{S}^{-1}\hat{S}(\hat{M}_j)_{\text{inp}}\hat{S}^t$$
$$= (S^t)^{-1}(\hat{M}_0)^{-1}_{\text{inp}}(\hat{M}_j)_{\text{inp}}\hat{S}^t, \tag{8}$$

and therefore

$$\text{tr}\big[(\hat{m}_1)_{\text{out}}(\hat{m}_2)_{\text{out}}\big]$$
$$= \text{tr}\big((S^t)^{-1}(\hat{M}_0)^{-1}_{\text{inp}}(\hat{M}_1)_{\text{inp}}\hat{S}^t(S^t)^{-1}(\hat{M}_0)^{-1}_{\text{inp}}(\hat{M}_2)_{\text{inp}}\hat{S}^t\big)$$
$$= \text{tr}\big((\hat{M}_0)^{-1}_{\text{inp}}(\hat{M}_1)_{\text{inp}}(\hat{M}_0)^{-1}_{\text{inp}}(\hat{M}_2)_{\text{inp}}\big)$$
$$= \text{tr}\big[(\hat{m}_1)_{\text{inp}}(\hat{m}_2)_{\text{inp}}\big]. \tag{9}$$

□

4 Properties

It was pointed out in Sect. 2 that the beam width and the far-field divergence take extreme values along the beam axes characterized, respectively, by the conditions $\langle xy\rangle = 0$ and $\langle uv\rangle = 0$. With regard to the intrinsic axes, it can be shown that *the quantities* $\text{tr}(\hat{m}_1^2)$ and $\text{tr}(\hat{m}_2^2)$ *reach their extreme values along the directions of such orthogonal axes*.

Proof Let us choose a new Cartesian coordinate system that makes an angle α with respect to the (arbitrary but fixed) laboratory axes. The former matrices $\hat{M}$, referred now to the rotated coordinate axes, $\hat{M}'$, would read

$$\hat{M}'_1 = \cos(2\alpha)\hat{M}_1 + \sin(2\alpha)\hat{M}_2, \tag{10a}$$
$$\hat{M}'_2 = -\sin(2\alpha)\hat{M}_1 + \cos(2\alpha)\hat{M}_2, \tag{10b}$$
$$\hat{M}'_0 = \hat{M}_0. \tag{10c}$$

Consequently, $\hat{m}'_1$ and $\hat{m}'_2$ become

$$\hat{m}'_1 = \cos(2\alpha)\hat{m}_1 + \sin(2\alpha)\hat{m}_2, \tag{11a}$$
$$\hat{m}'_2 = -\sin(2\alpha)\hat{m}_1 + \cos(2\alpha)\hat{m}_2, \tag{11b}$$

and then

$$\text{tr}\big(\hat{m}'_1\hat{m}'_2\big) = \cos(4\alpha)\,\text{tr}(\hat{m}_1\hat{m}_2) + \frac{\sin(4\alpha)}{2}\big[\text{tr}(\hat{m}_2^2) - \text{tr}(\hat{m}_1^2)\big] \tag{12a}$$

$$\text{tr}\big(\hat{m}'^2_1\big) = \cos^2(2\alpha)\,\text{tr}\big(\hat{m}_1^2\big) + \sin^2(2\alpha)\,\text{tr}\big(\hat{m}_2^2\big) + 2\sin(2\alpha)\cos(2\alpha)\,\text{tr}(\hat{m}_1\hat{m}_2) \tag{12b}$$

$$\text{tr}\big(\hat{m}'^2_2\big) = \sin^2(2\alpha)\,\text{tr}\big(\hat{m}_1^2\big) + \cos^2(2\alpha)\,\text{tr}\big(\hat{m}_2^2\big) - 2\sin(2\alpha)\cos(2\alpha)\,\text{tr}(\hat{m}_1\hat{m}_2) \tag{12c}$$

The extremal condition for $\text{tr}(\hat{m}_1^2)$ and $\text{tr}(\hat{m}_2^2)$ is

$$\frac{d[\text{tr}(\hat{m}'^2_1)]}{d\alpha} = \frac{d[\text{tr}(\hat{m}'^2_2)]}{d\alpha} = 0, \tag{13}$$

which finally gives

$$\text{tr}\big(\hat{m}'_1\hat{m}'_2\big) = 0. \tag{14}$$

In other words, $\text{tr}(\hat{m}_1^2)$ and $\text{tr}(\hat{m}_2^2)$ reach their extreme values when the laboratory axes and the intrinsic axes match. □

Equations (12a) and (14) can also be used for determining the rotation angle α between intrinsic and laboratory axes. We obtain at once

$$\tan(4\alpha) = \frac{2\,\text{tr}(\hat{m}_1\hat{m}_2)}{\text{tr}(\hat{m}_1^2) - \text{tr}(\hat{m}_2^2)}. \tag{15}$$

In order to get deeper insight into the physical meaning of $\text{tr}(\hat{m}_1^2)$ and $\text{tr}(\hat{m}_2^2)$ it will suffice to write these quantities at the waist plane in the form

$$\text{tr}\big(\hat{m}_1^2\big) = \big\langle\tilde{x}^2 - \tilde{y}^2\big\rangle^2 + \big\langle\tilde{u}^2 - \tilde{v}^2\big\rangle^2 + 2\langle\tilde{x}\tilde{u} - \tilde{y}\tilde{v}\rangle^2 \tag{16a}$$

and

$$\text{tr}\big(\hat{m}_2^2\big) = \tan^2(2\xi)\big\langle\tilde{x}^2 - \tilde{y}^2\big\rangle^2 + \tan^2(2\psi)\big\langle\tilde{u}^2 - \tilde{v}^2\big\rangle^2 + 2\langle\tilde{x}\tilde{v} + \tilde{y}\tilde{u}\rangle^2, \tag{16b}$$

with $\tilde{x} \equiv \frac{x}{\sqrt{\langle r^2\rangle}}$, $\tilde{y} \equiv \frac{y}{\sqrt{\langle r^2\rangle}}$, $\tilde{u} \equiv \frac{u}{\sqrt{\langle \eta^2\rangle}}$, $\tilde{v} \equiv \frac{v}{\sqrt{\langle \eta^2\rangle}}$, where the new variables are defined at the waist plane, and ξ and ψ refer to the angles defined by (2) and (3). It is clear from (16a) that, since $\text{tr}(\hat{m}_1^2)$ remains constant upon free propagation, this quantity provides a measure of the beam symmetry with respect to the laboratory axes. On the other hand, (16b) shows that $\text{tr}(\hat{m}_2^2)$ contains information about the beam rotation under propagation. When the field is referred to the intrinsic axes, both quantities, $\text{tr}(\hat{m}_1^2)$ and $\text{tr}(\hat{m}_2^2)$, would reach their extreme values. This property reminds the extreme values attained by the beam size and the far-field divergence along the beam axes defined by the conditions $\langle xy\rangle = 0$ and $\langle uv\rangle = 0$, respectively.

It is important to remark that the position of the intrinsic axes can be determined from the measurements of the second-order spatial moments that characterize any paraxial beam, even although the specific analytical field amplitude is not explicitly known. This possibility is a direct consequence of the definition of intrinsic axes by means of (7), along with the fact that $\hat{m}_1$ and $\hat{m}_2$ are expressed in terms of measurable irradiance moments.

All these properties make of practical use to choose the laboratory coordinate axes parallel to the intrinsic axes of the beam.

As a simple illustrative example, let us consider a partially coherent light field represented by the following stochastic process at plane $z = 0$ [18]:

$$E(r,\theta) = f(r)\exp(in\theta)\big[a\exp(-i\theta)+b\exp(i\theta)\big], \quad (n \neq 0) \tag{17}$$

where $f(r)$ is real, a and b are random variables, and (r,θ) denote planar polar coordinates. This expression means that the field $E(r,\theta)$ could be understood as a superposition of the spiral modes $n-1$ and $n+1$ (see, for example, [18–24]). For this field, the irradiance reads

$$I(r,\theta) = f^2(r)\big[\overline{a^2} + \overline{b^2} + 2\,\mathrm{Re}\big\{\overline{ab^*}\exp(-2i\theta)\big\}\big], \tag{18}$$

where the overbar symbolizes an ensemble average, and the asterisk indicates complex conjugation. For this field we have $\langle x\rangle = \langle y\rangle = \langle u\rangle = \langle v\rangle = 0$. Also note that the spatial profile of the beam can be shaped by fixing a particular function $f(r)$. The coherence between the modes can be expressed by the ratio [18]

$$\sigma = \frac{\overline{ab^*}}{(\overline{a^2 b^2})^{1/2}}, \tag{19}$$

which fulfills

$$0 \le |\sigma| \le 1, \tag{20}$$

where the value $\sigma = 0$ means that the modes are mutually uncorrelated (incoherence) and the beam is rotationally symmetric, whereas $|\sigma| = 1$ gives a coherent superposition of the modes.

Here we want to stress that any value $0 < |\sigma| \le 1$ has the remarkable property of that all three coordinate reference systems are identical at plane $z = 0$. In other words, the conditions $\langle xy\rangle = 0$, $\langle uv\rangle = 0$ and $\mathrm{tr}(\hat{m}_1\hat{m}_2) = 0$ generate the same orthogonal axes at plane $z = 0$. Thus, in the present example, $\xi = \psi = \alpha$ at $z = 0$. Of course, in general, this equality is no fulfilled.

It can also be shown that the angle α between the intrinsic axes and the particular laboratory axes we have used to write the field (cf. (17)) is given by

$$\tan 2\alpha = -\frac{\mathrm{Im}\{\langle ab^*\rangle\}}{\mathrm{Re}\{\langle ab^*\rangle\}}. \tag{21}$$

Let us finally recall that, when this field propagates through RSO systems, only the intrinsic axes would remain invariant.

5 Conclusions

A new set of transverse Cartesian axes has been introduced in terms of the second-order irradiance moments of a light beam. We call these orthogonal axes *the intrinsic axes of the beam* because they depend on measurable overall spatial characteristics of the field and, in addition, they remain invariant upon propagation through rotationally symmetric ABCD optical systems. Since the angle between these axes and the laboratory reference axes (arbitrary but fixed) can be determined from measurements of the second-order irradiance moments of the beam, this suggests the practical use of expressing any beam referred to their intrinsic axes.

Involving matrices $\hat{m}_1$ and $\hat{m}_2$, defined by (6), two quantities have also been found that reach extreme values along the directions parallel to the intrinsic axes. This resembles the well-known extreme values reached by the beam width and the far-field divergence along the transverse beam axes defined by the conditions $\langle xy\rangle = 0$ and $\langle uv\rangle = 0$, respectively. In the present case, these quantities are related, respectively, with the beam symmetry and with the rotation of the beam profile upon propagation. Note that all the results are valid for any partially coherent paraxial light beam.

It should finally be remarked that the present work uses a scalar framework. The vectorial case, involving polarization, and the recently introduced rotating light beams [25–28], which exhibit either rotating polarization or rotating transverse mode pattern, represent examples of fields that require a more general formalism and deserve study in the future.

Acknowledgements This work has been supported by the Ministerio de Ciencia e Innovación of Spain, project FIS2010-17543. One of the authors (A.M.) acknowledges a FPU grant from the Spanish Ministerio de Educación.

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