

# Beam width of highly-focused radially-polarized fields

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**Abstract:** On the basis of a formal analogy with the irradiance moments, analytical definitions are proposed for the width of both the transverse and the longitudinal component of rotationally-symmetric radially-polarized fields at the focal plane of a high-focusing optical system. The beam width of the whole field is also introduced. The transverse beam size is thus associated with the overall spatial structure of the field. The beam-width definitions are applied to an illustrative example, which enables us to show that, at the focal plane, the power contained within a circle whose radius is given by the proposed beam widths represents the main part of the total power.

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## 1. Introduction

In nanophotonics and nanooptics, a large number of optical devices make use of highly-focused light beams (see, for example, Ref. 1 and references therein). Applications range from optical data storage to confocal microscopy and particle trapping, to mention only some

examples. In these cases, the spatial distribution of the polarization of the incident field has been shown to play an essential role. More specifically, radially polarized beams can be particularly useful because, at their focal region, they exhibit spot sizes smaller than the widths shown by conventional linearly polarized fields (see, for instance, Refs. 2-4). In the experiments, the spot size was introduced as the area that is encircled by the contour line at half the maximum value of the irradiance profile. Consequently, in such framework, the beam width is associated with a local value of the irradiance.

For paraxial fields, however, the concept of transverse beam size is closely related with the power-content ratio inside the spot radius. Accordingly, the beam size involves the overall spatial structure over a transverse plane, which is described by means of the so-called irradiance-moments formalism [5–13]. This characterization is particularly valuable in those cases in which the light fields show complicated and irregular spatial structures. If the electric-field vector of a paraxial light beam is denoted by  $\mathbf{E}(r, \phi) = (E_x, E_y)$ ,  $r$  and  $\phi$  being the planar polar coordinates, the (squared) beam width is then determined by the second-order irradiance moment

$$\langle x^2 + y^2 \rangle \equiv \langle r^2 \rangle = \frac{\iint r^2 |\mathbf{E}|^2 r dr d\phi}{I} = \frac{I_x \langle r^2 \rangle_x}{I} + \frac{I_y \langle r^2 \rangle_y}{I}, \quad (1)$$

where

$$\langle r^2 \rangle_j \equiv \frac{\iint r^2 |E_j|^2 r dr d\phi}{I_j}, \quad j = x, y, \quad (2)$$

$$I_j \equiv \iint |E_j|^2 r dr d\phi, \quad j = x, y, \quad (3)$$

and  $I = I_x + I_y$ , the integrals being calculated over the entire transverse plane. The usefulness of this definition is confirmed because it is currently adopted as ISO standard. The difficulty arises, however, when one tries to extend this definition of beam width to highly-focused fields due to convergence problems of the integrals. In fact, to our knowledge, no analytical overall parameter has yet been proposed for representing the transverse beam size of light fields at the region where the power is highly concentrated. This is just the aim of the present work. It should be remarked that the vectorial character of light force to introduce, in a separate way, beam-width definitions for both the transverse and the longitudinal components of the field.

The paper is organized as follows. In the next Section, we introduce the notation used in this work along with the analytical definition proposed for the beam width of the transverse component of the field. Attention will be devoted to the important case of radially polarized fields impinging on a high-focusing system. The beam width of the longitudinal component is considered in Section 3, and the transverse size of the global beam is given in Section 4. In all the cases the beam width is expressed in terms of the filling factor and the angular aperture of the focusing optical device. Section 5 remarks a meaningful relation between power content and beam width. The definitions are applied to an illustrative example in Section 6, which analyzes depleted-center beams. Finally, the main conclusions are summarized in Section 7.

## 2. Width of the transverse component of the field at the focal plane

Let us consider a monochromatic radially polarized field at the input plane of an aplanatic focusing optical system. As is well known [1], the electric field vector  $\mathbf{E}$  at the focal plane of the system is given by the following expression:

$$\mathbf{E}(r, \phi) = A \int_0^{\theta_0} \int_0^{2\pi} E_i(\theta) \sqrt{\cos \theta} \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix} \exp[ikr \sin \theta \cos(\phi - \varphi)] \sin \theta d\theta d\varphi, \quad (4)$$

where  $A$  is a constant,  $E_i$  is the field amplitude (incident on the system) assumed to be rotationally symmetric around the propagation axis  $z$ ;  $r$  and  $\phi$  denote here the polar coordinates at the focal plane, and the angles  $\theta$  and  $\theta_0$  are represented in Fig. 1. In Eq. (4) the vector inside the integral provides the vectorial structure of a pure incident radially polarized field, and the value  $z = 0$  corresponds to the focal plane where we are evaluating  $\mathbf{E}(r, \phi)$ .

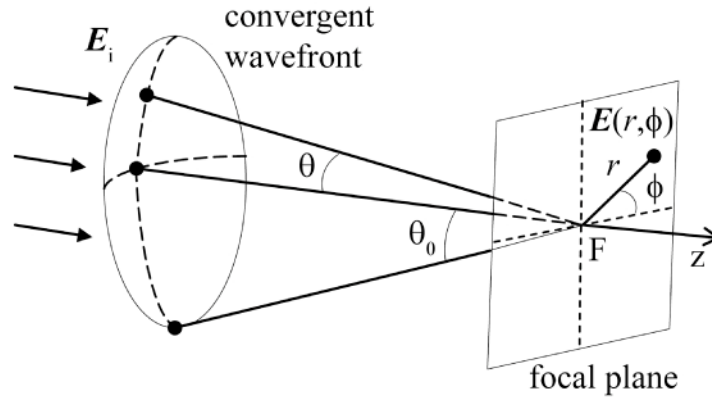


Fig. 1. Illustrating the notation and the geometry of the problem. The value  $\theta_0$  corresponds to the semi-aperture angle of the aplanatic system.

For simplicity, let us write

$$\rho \equiv \sin \theta. \quad (5)$$

Accordingly, the transverse and longitudinal field-components at the focal plane become

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = A \int_0^a \int_0^{2\pi} E_i(\rho) (1 - \rho^2)^{1/4} \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \exp[ikr \rho \cos(\phi - \varphi)] \rho d\rho d\varphi, \quad (6.a)$$

$$E_z = -A \int_0^a \int_0^{2\pi} \frac{\rho E_i(\rho)}{(1 - \rho^2)^{1/4}} \exp[ikr \rho \cos(\phi - \varphi)] \rho d\rho d\varphi, \quad (6.b)$$

where  $a \equiv \sin \theta_0$  takes into account the angular aperture of the focusing system. For convenience, in the rest of the paper the incident field amplitude  $E_i$  will be written in the form

$$E_i(\rho) = \rho h(\rho), \quad (7)$$

where  $h(\rho)$  denotes an arbitrary rotationally-symmetric function.

In the present section we will investigate the transverse part of the field at the focal plane. Note first that Eq. (6.a) can be written in the form

$$\mathbf{E}_T \equiv \begin{pmatrix} E_x(r, \phi) \\ E_y(r, \phi) \end{pmatrix} = 2\pi i A \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} f(r), \quad (8)$$

with

$$f(r) = \int_0^a E_i(\rho)(1-\rho^2)^{1/4} J_1(kr\rho) \rho d\rho, \quad (9)$$

where we have used the relation

$$\int_0^{2\pi} \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} \exp[ikr\rho \cos(\phi - \varphi)] d\varphi = 2\pi i \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix} J_1(kr\rho). \quad (10)$$

Thus, the squared modulus of the transverse component (proportional to the irradiance), at the focal region, reads  $|\mathbf{E}_T|^2 = |E_x|^2 + |E_y|^2 = 4\pi^2 |A|^2 |f(r)|^2$ , so that

$$I_T(r) \equiv \int_0^{2\pi} |\mathbf{E}_T(r, \phi)|^2 d\phi = (2\pi)^3 |A|^2 |f(r)|^2. \quad (11)$$

On the basis of the formal analogy with the paraxial case (cf. Eq. (1)), we see that the width of the transverse component (symbolized by  $w_T$ ) is closely connected with the width of function  $f(r)$ . If we try to apply the second-order moment definition to the width of function  $f$ , we would find that, in general, the integral diverges. To overcome this trouble and introduce a general width definition for this function, we write  $f(r)$  in the form

$$f(r) = f_1(r) + f_2(r), \quad (12)$$

where

$$f_1(r) = \frac{a^2}{kr} F(a) J_2(kra), \quad (13.a)$$

$$f_2(r) = -\frac{1}{kr} \int_0^a \rho^2 J_2(kr\rho) \frac{\partial F(\rho)}{\partial \rho} d\rho, \quad (13.b)$$

with

$$F(\rho) = h(\rho)(1-\rho^2)^{1/4}. \quad (14)$$

It should be noted that, in a preliminary work, recently reported [14], we wrote function  $f(r)$  in an alternative way. Here we have chosen the analytical expressions given by Eqs. (13) because they will be easier to use for studying the beam-width changes of the focused field at transverse planes around the focal region.

In a similar way to Eq. (1),  $w_T^2$  can formally be defined as follows:

$$w_T^2 \equiv \frac{w_{T1}^2 I_1 + w_{T2}^2 I_2}{I_T}, \quad (15)$$

where  $w_{T1}^2$  and  $w_{T2}^2$  symbolize here the (squared) widths associated to  $f_1$  and  $f_2$ ,

$$I_T \equiv \int_0^\infty \int_0^{2\pi} |\mathbf{E}_T(r, \phi)|^2 r dr d\phi = \int_0^\infty I_T(r) r dr = \frac{8\pi^3}{k^2} |A|^2 \int_0^a \left| \rho h(\rho)(1-\rho^2)^{1/4} \right|^2 \rho d\rho, \quad (16)$$

and

$$I_j = (2\pi)^3 |A|^2 \int_0^\infty |f_j(r)|^2 r dr, \quad j = 1, 2. \quad (17)$$

Note that, to obtain Eq. (16), use of Parseval theorem has been made. From Eq. (15), we see that the problem of establishing an extended definition of width for this kind of highly-focused vectorial fields reduces to defining, in a suitable way, the quantities  $w_{T1}^2$  and  $w_{T2}^2$ .

Concerning  $w_{T2}^2$ , this width can easily be defined as a second-order moment, namely,

$$w_{T2}^2 = \frac{(2\pi)^3 |A|^2 \int_0^\infty r^2 |f_2(r)|^2 r dr}{I_2} = \frac{(2\pi)^3 |A|^2}{k^4 I_2} \int_0^a \rho^2 \left| \frac{\partial F(\rho)}{\partial \rho} \right|^2 \rho d\rho. \quad (18)$$

However, if one tries to apply the above moment definition to  $w_{T1}^2$ , it can be shown that the integral diverges. An alternative definition should then be provided. To do this note first that

$$|f_1(r)|^2 = a^6 |F(a)|^2 \left[ \frac{J_2(kra)}{kra} \right]^2. \quad (19)$$

As occurs, for instance, in diffraction theory, the transverse size of this function could be characterized by the position of the second zero of the Bessel function  $J_2$ . We denote this value  $c_2$ . Thus, we have

$$I_1 w_{T1}^2 = \frac{(2\pi)^3 |A|^2}{4k^4} a^2 c_2^2 |F(a)|^2, \quad (20)$$

where we have used the equality

$$\int_0^\infty |f_1(r)|^2 r dr = a^6 |F(a)|^2 \int_0^\infty \left[ \frac{J_2(kra)}{kra} \right]^2 r dr = \frac{a^4}{4k^2} |F(a)|^2. \quad (21)$$

In summary, the (squared) width  $w_T^2$  of the transverse field takes the analytical form

$$w_T^2 = \frac{1}{k^2 I_{0T}} \left[ \frac{a^2 c_2^2}{4} |F(a)|^2 + \int_0^a \rho^2 \left| \frac{\partial F(\rho)}{\partial \rho} \right|^2 \rho d\rho \right], \quad (22)$$

where

$$I_{0T} = \int_0^a \left| \rho h(\rho) (1 - \rho^2)^{1/4} \right|^2 \rho d\rho. \quad (23)$$

Two special cases should be remarked, which correspond to vanishing either  $f_1$  or  $f_2$  in Eq. (12):

- a) For those highly-focused fields whose function  $F(\rho)$  equals zero at  $\rho = a$ ,  $w_T^2$  reduces to the well-known paraxial expression.
- b) When  $h(\rho)(1 - \rho^2)^{1/4} = \text{constant} = h_0$ , then  $\frac{\partial F}{\partial \rho} = 0$ , so that  $f_2(r) = 0$ . Thus we obtain the simple expression

$$w_T^2 = \frac{h_0^2}{k^2} \frac{a^6 c_2^2}{16} |F(a)|^2. \quad (24)$$

This case corresponds to an incident beam whose amplitude  $E_i$  reads

$$E_t(\rho) = \rho h(\rho) = \frac{h_0 \rho}{(1-\rho^2)^{1/4}}. \quad (25)$$

### 3. Width of the longitudinal component of the field at the focal plane

For the sake of convenience, let us write the amplitude of the longitudinal component,  $E_z(r, \phi)$ , in the form (see Eq. (6.b))

$$E_z(r, \phi) = -2\pi A \int_0^a G(\rho) J_0(kr\rho) \rho d\rho, \quad (26)$$

where

$$G(\rho) = \frac{\rho^2 h(\rho)}{(1-\rho^2)^{1/4}}. \quad (27)$$

By applying a similar procedure to that used in the previous section with regard to the transverse field, we define  $w_L^2$  as follows:

$$w_L^2 = \frac{1}{k^2 I_{0L}} \left[ \frac{c_1^2}{2} |G(a)|^2 + \int_0^a \left| \frac{\partial G(\rho)}{\partial \rho} \right|^2 \rho d\rho \right], \quad (28)$$

where  $c_1$  denotes the first zero of the Bessel function  $J_1$ , and

$$I_{0L} = \int_0^a \left| \frac{\rho^2 h(\rho)}{(1-\rho^2)^{1/4}} \right|^2 \rho d\rho. \quad (29)$$

### 4. Global width of the whole beam

Let us now consider the whole beam, whose irradiance is proportional to the global (squared) amplitude, i.e.,  $|\mathbf{E}|^2 = |E_x|^2 + |E_y|^2 + |E_z|^2$ . On the basis of the analogy with the second-order irradiance moments (see Eqs. (1-3)), we can introduce  $w_G^2$  for the whole field in the form

$$w_G^2 = \frac{I_T w_T^2 + I_L w_L^2}{I}, \quad (30)$$

where  $I_T$  is given by Eq. (16), and

$$I_L \equiv \int_0^a \int_0^{2\pi} |E_z(r, \phi)|^2 r dr d\phi, \quad (31)$$

with  $I_G = I_T + I_L$ . From the beam-width definitions introduced in the above sections for the field components, we finally get

$$w_G^2 = \frac{1}{k^2 I_{0G}} \left\{ \frac{a^2 c_2^2}{4} |F(a)|^2 + \frac{c_1^2}{2} |G(a)|^2 + \int_0^a \left[ \rho^2 \left| \frac{\partial F(\rho)}{\partial \rho} \right|^2 + \left| \frac{\partial G(\rho)}{\partial \rho} \right|^2 \right] \rho d\rho \right\}, \quad (32)$$

where  $I_{0G} = I_{0T} + I_{0L}$ , with  $I_{0T}$  and  $I_{0L}$  defined by Eqs. (23) and (29).

## 5. Power-content ratio

The validity of the proposed definitions for the beam width should be tested by evaluating the radius of the region (around the z-axis) where the power is concentrated. High enough values of the ratio

$$P_{wj} = \frac{\int_0^{w_j} I_j(r) r dr}{\int_0^{\infty} I_j(r) r dr}, \quad j = T, L, G, \quad (33)$$

would confirm the appropriateness of the analytical definitions. But before applying them to an illustrative example, an important remark should be pointed out by making use of the Tchebycheff's inequality [15].

Let us consider the simple paraxial (scalar) case, and denote by  $I(r)$  and  $\langle r^2 \rangle$  the (rotationally symmetric) irradiance distribution of the beam profile and its associated second-order moment, respectively. It follows at once

$$\int_0^{\infty} r^2 I(r) r dr \geq \int_{R_0}^{\infty} r^2 I(r) r dr \geq R_0^2 \int_{R_0}^{\infty} I(r) r dr, \quad (34)$$

where  $R_0$  is an arbitrary positive number. We then have

$$\frac{\int_0^{R_0} I(r) r dr}{\int_0^{\infty} I(r) r dr} = 1 - \frac{\int_{R_0}^{\infty} I(r) r dr}{\int_0^{\infty} I(r) r dr} \geq 1 - \frac{1}{R_0^2} \frac{\int_0^{\infty} r^2 I(r) r dr}{\int_0^{\infty} I(r) r dr} = 1 - \frac{\langle r^2 \rangle}{R_0^2}. \quad (35)$$

Now, in order to assure that, at least, 75% of the total power is contained within a circular region around the z-axis, we set

$$\frac{\int_0^{R_0} I(r) r dr}{\int_0^{\infty} I(r) r dr} \geq 1 - \frac{\langle r^2 \rangle}{R_0^2} > \frac{3}{4}, \quad (36)$$

and, therefore,

$$R_0 \geq 2\sqrt{\langle r^2 \rangle}. \quad (37)$$

In other words, the radius  $R_0$  of the region that assures 75% of the total power should be twice the root of the second-order irradiance moment.

Taking this into account, in the next section we adopt a formal analogy with Eq. (37), and the power-content ratio will be computed in the examples by integrating within a circular region whose radius is twice the specific beam width (transverse, longitudinal or global). In other words, we will calculate the ratio

$$P_{2w_j} = \frac{\int_0^{2w_j} I_j(r) r dr}{\int_0^{\infty} I_j(r) r dr}, \quad j = T, L, G. \quad (38)$$

## 6. Application to an example

Let us now make use of the above analytical definitions, and consider the particular but illustrative case of incident depleted-center field amplitude  $E_i$  given by

$$E_i(\rho) = \rho h(\rho) = \rho \exp\left(-\frac{f^2 \rho^2}{\omega_0^2}\right) = \rho \exp\left(-\frac{\rho^2}{f_0^2 a^2}\right), \quad (39)$$

where  $f_0$  represents the so-called filling factor, which, for this field, can be defined in the usual way [1], namely,  $f_0 = \omega_0 (f \sin \theta_0)^{-1}$ ,  $f$  being the focal length of the focusing system. Figure 2 plots the irradiance distributions (with  $f_0 = 1$ ) of the field components and of the whole beam, calculated at the focal plane of the high-focusing system in terms of the radial distance to the  $z$ -axis (the field is rotationally symmetric). It is clear that the irradiance is significant for radial distances smaller than  $\lambda$ .

Figure 3 shows (a proportionality factor  $k$  apart) the widths  $2w_T$ ,  $2w_L$  and  $2w_G$  in terms of the filling factor. According with the discussion of Section 5, we have computed the value  $2w$  to assure a significant enough power-content ratio inside a circle whose radius is  $2w$  (see also Fig. 4). It should also be noted that the three curves take the same value for a certain  $f_0$ . It should also be noted that, although the initial beam exhibits a depleted-center behavior, the spatial profile of the global beam at the focal region shows a bell-shaped structure.

Finally, Fig. 4 enables to test whether the proposed definitions for the width are physically consistent with the size of the region where the power is concentrated. More specifically, this figure provides the ratio

$$P_{2w_j} = \frac{\int_0^{2w_j} I_j(r) r dr}{\int_0^{\infty} I_j(r) r dr}, \quad j = T, L, G. \quad (40)$$

Note that abscises use the same scale in Figs. 3 and 4. As it should be expected, we see from the figure that  $P_{2w}$  is always higher than 88% of the total power. In other words, light energy is mainly focused on a circular region around the  $z$ -axis whose radius is twice the respective beam width. It is also interesting to note the small range of variation (less than 10%) of the power-content ratio  $P_{2w}$  for large variations of the filling factor ( $f_0 \in [0.5, 1.5]$ ).



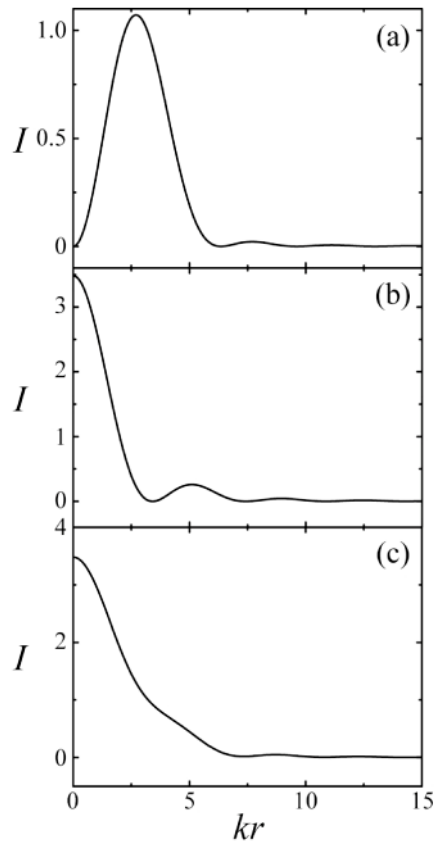


Fig. 2. Irradiance (arbitrary units) at the focal plane versus the radial distance  $r$  to the propagation axis  $z$  (in the figures, the curves have been represented in terms of the dimensionless parameter  $kr$ ). The figures refer to the transverse component (Fig. 2.a), the longitudinal component (Fig. 2.b), and the whole field (Fig. 2.c). In all the figures, the same scale has been used for ordinates, and the filling factor  $f_0$  has been chosen equal to 1.

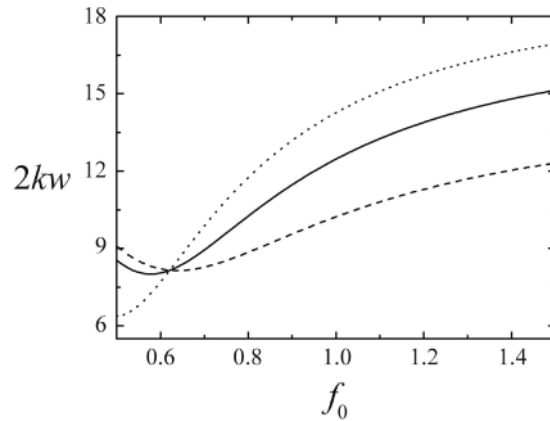


Fig. 3. Dimensionless parameter  $2kw$  at the focal plane, associated with the transverse component (dashed line,  $2kw_T$ ), the longitudinal component (dotted line,  $2kw_L$ ), and the whole field (continuous line,  $2kw_G$ ) versus the filling factor  $f_0$ .

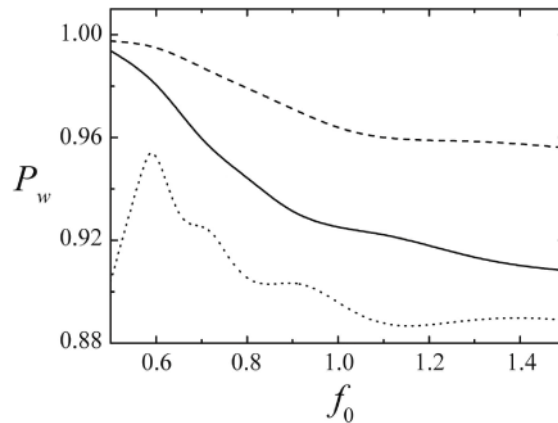


Fig. 4. Power-content ratio  $P_{2w}$  (defined by Eq. (40)) in terms of  $f_0$ . Dashed line, dotted line and continuous line correspond, respectively, to the transverse component, longitudinal component and whole field.

## 7. Conclusions

Associated with the overall spatial structure of a vectorial field, the beam-width definitions based on the irradiance moments have been extended from the paraxial case to highly-focused vectorial fields. More specifically, for incident radially-polarized beams impinging on an aplanatic focusing system, analytical definitions have been proposed for characterizing the beam width of both the transverse and the longitudinal components of the field at the region (focal plane) where the light power is highly concentrated. From the combination of the above definitions, the beam width of the whole field can also be introduced.

The power-content ratio within a circle whose radius is given by the beam width has been investigated by means of an illustrative example, concerning incident radially-polarized depleted-center fields. The results confirm the suitability of the proposed definitions to analytically characterize the widths associated with highly-focused fields at the focal plane. The evolution of the beam widths upon propagation from this plane constitutes the next step of this research and deserves further study in the future.

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